DESIGN AND ANALYSIS OF MODEL PREDICTIVE CONTROL USING MPT TOOLBOX

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Abstract

The Multi-Parametric Toolbox (MPT) is a useful toolbox for Matlab that helps by computing explicit optimal control law for constrained systems. The recent concept of Model Predictive Control (MPC) that ensures closed-loop stability uses the polytopic or ellipsoidal invariant sets. The MPT toolbox provides a number of algorithms for computing with polytopic sets. In this paper, the MPT toolbox is used for designing of MPC controller that guarantees closed-loop stability and asymptotic reference tracking.

Keywords: Predictive control; Tracking; Invariance; Polyhedral set; Linear programming, Quadratic programming.

1 Introduction

The recent concept of the model predictive control (MPC) with guaranteed stability for the linear time-invariant (LTI) systems uses the positive invariant sets (Gilbert and Tan, 1991). The invariant set is the target for the system state trajectory on a given length of prediction horizon. For each prediction horizon length, the maximal feasible set of initial system states can be found. This approach is know as a dual-mode paradigm (Rossiter, 2003; Mayne *et al.*, 2000) in which the linear controller ensures the system stability and constraints satisfaction over the second (infinite horizon) mode. Over the first (finite horizon) mode, the perturbations to the linear control law are added to ensure the constraints satisfaction.

The Multi-Parametric Toolbox (MPT) for Matlab developed by (Kvasnica *et al.*, 2003) is a tool that simplifies the design and analysis of model based predictive controllers that use the concept of invariant sets. In standard MPC, the optimization problem (quadratic program) is solved on-line at each sampling interval but such approach is not suitable for the fast systems. For fast systems, it is possible to solve the optimization problem explicitly off-line (Gal, 1995; Gal and Nedoma, 1972; Bemorad *et al.*, 2002). The MPT toolbox provides the solvers for computing the explicit optimal feedback control laws for constrained linear and piecewise affine systems. It also provides a number of algorithm for computing with polytopic sets and the functions for their visualization. The visualization is helpful namely by analysis of designed MPC.

2 Stabilizing Model Predictive Control

Consider a state space model of discrete LTI system in the form

$$x_{k+1} = Ax_k + Bu_k \quad (1)$$

$$y_k = Cx_k + Du_k \ . \quad (2)$$

with constraints on the control variable and the state in the form

$$u_{\min} \leqslant u_k \leqslant u_{\max} \ , \quad x_{\min} \leqslant M_x x_k \leqslant x_{\max} \ . \tag{3}$$

The prediction horizon was splitted into two intervals. The constraints satisfaction over the mode 1 is ensured by solving the QP problem with constraints (3). In mode 2 ($k = N, N+1, \ldots$), the system is controlled under the feedback law $u_k = -Kx_k$ and the constraints (3) can be equivalently expressed as

$$\begin{bmatrix} u_{\min} \\ x_{\min} \end{bmatrix} \leqslant \begin{bmatrix} -K \\ M_x \end{bmatrix} x_k \leqslant \begin{bmatrix} u_{\max} \\ x_{\max} \end{bmatrix}, \quad k = N, N+1, \ldots \tag{4}$$

For the constraints (4), the following notation will be used in the next text

$$m_{\min} \leqslant M x_k \leqslant m_{\max} \ . \tag{5}$$

An efficient robust MPC (ERMPC) algorithm was presented in (Kouvaritakis *et al.*, 2000). This algorithm is based on dual mode approach. It is divided into two parts: off-line and on-line. The off-line part computes an ellipsoidal robust invariant set (LMI problem). The on-line part is based on a simple root finding problem, so QP or SDP does not need to be used. In this algorithm the closed-loop paradigm is used where the control law is

$$u_k = -Kx_k + c_k \tag{6}$$

and the state prediction equation is given by the state space model

$$x_{k+1} = (A - BK)\, x_k + Bc_k \ . \tag{7}$$

Only a finite number of the sequence c_k is non-zero - over the mode 1. The length of the prediction horizon over the mode 1 will be denoted as N ($k = 0, 1, \ldots, N-1$), so $c_N = 0$, $c_{N+1} = 0, \ldots$. At times $k = N+1, N+2, \ldots$, the state will be inside the invariant set and the control law will be $u_k = -Kx_k$ (mode 2).

In the ERMPC, the ellipsoidal invariant sets were used. It should be noted, that the ellipsoidal sets are only approximations for the linear systems. For linear systems, the nature shape of invariant set is an polytope. In the following sections, we will use the similar ideas as in ERMPC but we will consider the polytopic invariant sets.

2.1 Augmented State Space Model

One possible way to ensure the constraints satisfaction for the control variable is to define an augmented state vector z_k

$$z_k = \begin{bmatrix} x_k^T & c_k^T & \ldots & c_{k+N-1}^T \end{bmatrix}^T \ . \tag{8}$$

Let us denote the sequence

$$\begin{bmatrix} c_k & \ldots & c_{k+N-1} \end{bmatrix} = \vec{c}_k$$

Then the prediction dynamics will be

$$z_{k+1} = \tilde{\Phi}_i z_k \ , \qquad u_k = \tilde{K} z_k \tag{9}$$

where

$$\tilde{\Phi} = \begin{bmatrix} A - BK & B & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{K} = \begin{bmatrix} -K & I & 0 & \ldots & 0 \end{bmatrix} .$$

It is possible to invoke the feasibility of control variable implicitly by the use of the concept of invariance at current time and therefore the next step is to find the polytopic invarinat set for the augmented state space model.

2.2 Polytopic Invariant Set for Augmented Model

Let us define an augmented polytopic set for augmented state space model in the form

$$\Omega = \{z;\ M_\Omega z \leqslant m_\Omega\} \tag{10}$$

The feasibility conditions (3 - 5) become the following form

$$\begin{bmatrix} u_{\min} \\ x_{\min} \end{bmatrix} \leqslant \begin{bmatrix} \tilde{K} \\ M_x \begin{bmatrix} I & 0 & \ldots & 0 \end{bmatrix} \end{bmatrix} z_k \leqslant \begin{bmatrix} u_{\max} \\ x_{\max} \end{bmatrix} \tag{11}$$

that can be written as

$$m_{min} \leqslant M z_k \leqslant m_{max}\ . \tag{12}$$

The augmented invariant set (10) that respect the constraints (12) can be determined by the algorithm presented in (Gilbert and Tan, 1991).

The augmented invariant sets contains the following two important infirmations

1. *Invariant set* Ω^x: The invariant set for mode 2, where the control law is $u_k = -Kx_k$. If the state enters into this invariant set, it remains inside at the future time and the constraints will not be violated.

2. *Invariant set* Ω^c: The maximal polytopic subspace of the state space of initial conditions for a given prediction horizon where the control law is $u_k = -Kx_k + c_k$ and we are able to find the sequence $\vec{c}_k$ so that the constraints are not violated and the end of the state trajectory (at the end of prediction horizon) falls into the invariant set for mode 2.

The presented concept ensures the closed loop stability. Next question is how to find the both invariant sets Ω^x and Ω^c. The first one can be found by setting the sequence $\vec{c}_k = 0$. The augmented state vector will be

$$z_k = \begin{bmatrix} x_k \\ \vec{c}_k \end{bmatrix} = \begin{bmatrix} x_k \\ 0 \end{bmatrix} \tag{13}$$

and therefore the augmented invariant set is

$$M_\Omega z_k = \begin{bmatrix} M_\Omega^x & | & M_\Omega^c \end{bmatrix} \begin{bmatrix} x_k \\ \vec{c}_k \end{bmatrix} \leqslant m_\Omega\ . \tag{14}$$

The polytopic invariant set for mode 2 control is given by

$$\Omega^x = \{x;\ M_\Omega^x x \leqslant m_\Omega\}\ . \tag{15}$$

Now, we should find the second invariant set which is the set of feasible initial conditions Ω^c for a given length of prediction horizon. This set is given by the projection of the augmented invariant set Ω onto the state space. Some algorithms for projection of polytopic sets are implemented in MPT toolbox.

2.3 Final MPC algorithm

The final MPC algorithm that guarantees stability of the closed loop is the solution of the standard QP problem with constraints (14) for a given x_k, also

$$\vec{c}_k^{\,*} = \arg\min_{\vec{c}_k} J_c \qquad \text{s.t.} \quad M_\Omega^c \vec{c}_k \leqslant m_\Omega - M_\Omega^x x_k\ . \tag{16}$$

This QP is solved on-line in each sampling interval. With this approach, the stability of closed loop is guaranteed.

2.4 Examples

In this section, we present an example of augmented invariant set. Consider the second order discreet-time system described by a state space model

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k \tag{17}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \tag{18}$$

In this example, we used the following weighting matrices

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1 \;.$$

and constraints

$$\begin{bmatrix} -20 \\ -20 \end{bmatrix} \leqslant x \leqslant \begin{bmatrix} 20 \\ 20 \end{bmatrix}, \qquad -2 \leqslant u \leqslant 2 \;.$$

The invariant sets for different prediction horizons are show in the Fig.1. For each prediction horizon, the inner polytopic set (red one) is the invariant set for the mode 2 where the control law $u_k = -Kx_k$ does not violate the given constraints. The set of all possible initial states for a given prediction horizon is defined by the bigger polytopic set.

The Fig.2 shows an example of control for the initial condition $x_0 = [19.59\ ; -10.08]^T$ and prediction horizon $N = 15$.

In the Fig.3, the invariant sets for prediction horizon $N = 1, \ldots, 15$ and two different weighting matrices ($Q = I$ and $Q = 100 \cdot I$) are compared.

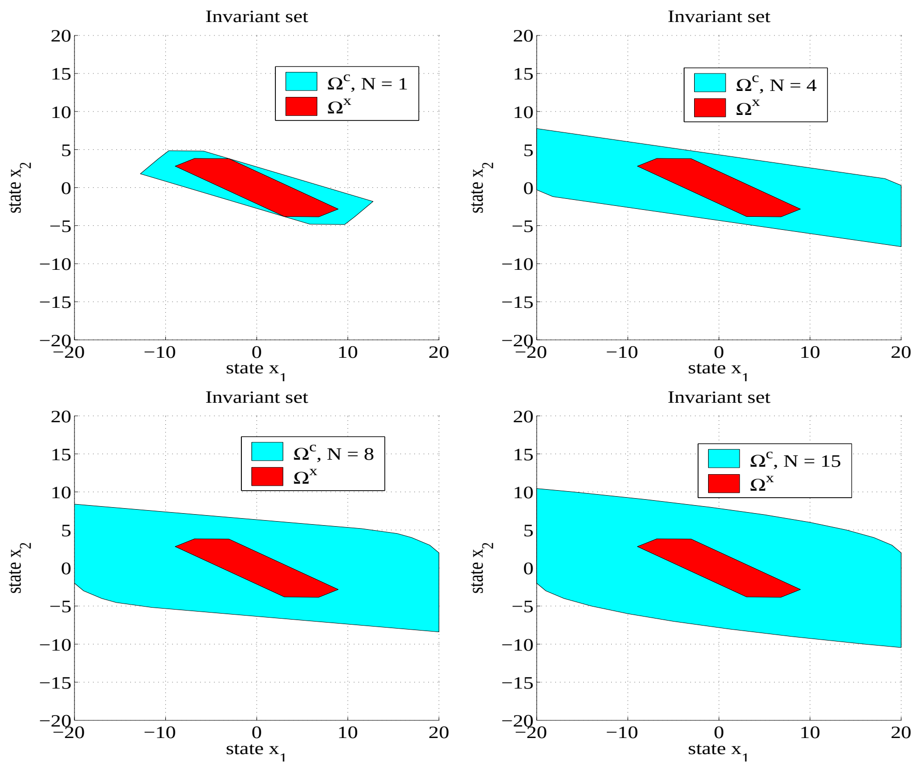

Figure 1: Invariant sets for prediction horizon $N = 1, 4, 8, 15$.

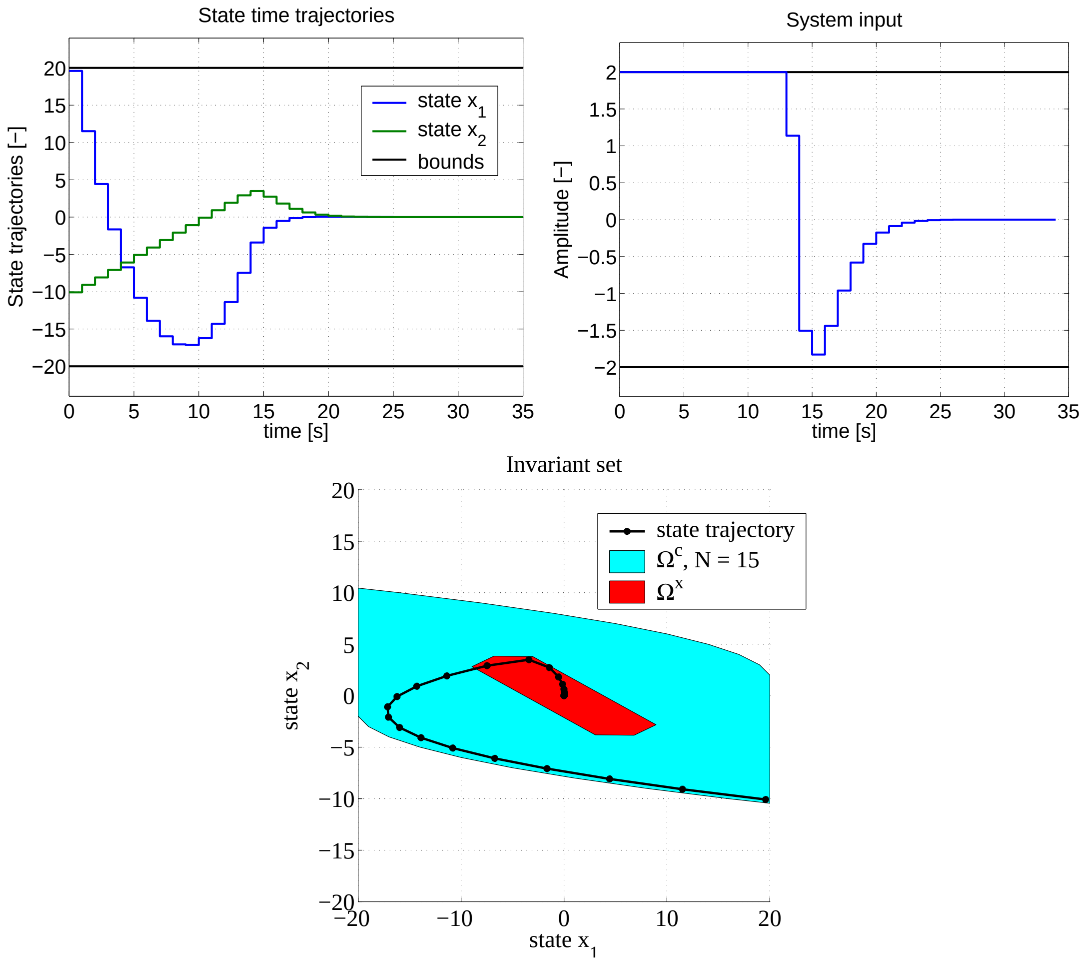

Figure 2: An example of state trajectory for initial condition $x_0 = [19.59\ ; -10.08]^T$ and prediction horizon $N = 15$.

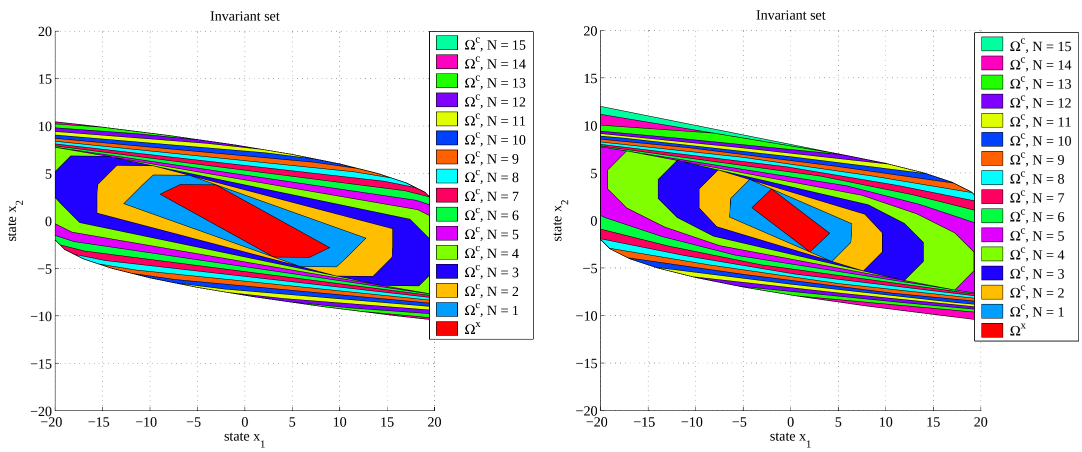

Figure 3: Invariant sets for prediction horizons $N = 1, 2, \ldots, 15$. Left: $Q = I$, Right: $Q = 100 \cdot I$.

3 Reference Tracking

The basic algorithm of the stabilizing MPC was introduced in the previous sections. In control applications, we usually want the system output to follow a given reference signal and therefore, in this section, we will show the extension of basic algorithm for constant reference tracking. Consider that the reference signal can be described by the state space model of the form

$$r_{k+1} = A_r r_k \tag{19}$$
$$w_k = C_r r_k \tag{20}$$

where

$$w_{\min} \leqslant w_k \leqslant w_{\max} \ . \tag{21}$$

We can introduce the following augmented state space model

$$\widehat{x}_{k+1} = A_m \widehat{x}_k + B_m u_k \tag{22}$$
$$\widehat{e}_k = C_m \widehat{x}_k \tag{23}$$

where

$$\widehat{x}_k = \begin{bmatrix} x_k \\ r_k \end{bmatrix}, \ A_m = \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \ B_m = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ C_m = \begin{bmatrix} C & -C_r \end{bmatrix} \ .$$

The constraints of the augmented model (22) are given by (3) and (21). The reference tracking problem was transformed to the standard problem which can be solved by algorithm presented in Section 2. The augmented invariant set (14) has now the form

$$M_\Omega z_k = \begin{bmatrix} M_\Omega^{\widehat{x}} & M_\Omega^{c} \end{bmatrix} \begin{bmatrix} \widehat{x}_k \\ \vec{c}_k \end{bmatrix} \leqslant m_\Omega \ . \tag{24}$$

For each prediction horizon N, the projection of previous invariant set onto the state space for the model (22) can be found. Note that the reference signal is included in the state space. This projection was denoted as Ω^c and is given by

$$M_\Omega^{\widehat{x}_0} \widehat{x} \leqslant m_\Omega^{\widehat{x}_0} \ .$$

The set of feasible initial conditions Ω^c usually does not cover all possible values of reference signal defined by (21)), namely for the short prediction horizons. Therefore, we have to find the minimal and maximal value of the reference signal for each x from invariant set Ω^c. This can be done by solving the linear programs

$$\widehat{x}_{k_{\max}} = \arg\max_{\widehat{x}} \left\{ \begin{bmatrix} 0 & C_r \end{bmatrix} \widehat{x}; \ M_\Omega^{\widehat{x}_0} \widehat{x} \leqslant m_\Omega^{\widehat{x}_0}, \ \begin{bmatrix} I & 0 \end{bmatrix} \widehat{x} = x_k \right\} \tag{25}$$
$$\widehat{x}_{k_{\min}} = \arg\min_{\widehat{x}} \left\{ \begin{bmatrix} 0 & C_r \end{bmatrix} \widehat{x}; \ M_\Omega^{\widehat{x}_0} \widehat{x} \leqslant m_\Omega^{\widehat{x}_0}, \ \begin{bmatrix} I & 0 \end{bmatrix} \widehat{x} = x_k \right\} \ . \tag{26}$$

The previous problem can be also solved as a multi-parametric linear program in MPT toolbox (without equality constraints). The result of such optimization problem are the regions in the state space where for each region, the optimizer ($w_{k_{\max}}$ and $w_{k_{\min}}$) is given by affine function of state x.

3.1 Example

In this section, we will show an example of piecewise constant reference tracking. Consider the dynamical system with constraints as in Section 2.4

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_k$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \ .$$

$$\begin{bmatrix} -20 \\ -20 \end{bmatrix} \leqslant x \leqslant \begin{bmatrix} 20 \\ 20 \end{bmatrix}, \qquad -2 \leqslant u \leqslant 2 \ .$$

The state space representation of piecewise constant signal w_k can be written as

$$\begin{aligned} r_{k+1} &= r_k \\ w_k &= r_k \end{aligned}$$

The piecewise constant signal w_k is constrained by

$$-15 \leqslant w_k \leqslant 15 \ .$$

The final augmented state space description where the model of reference signal (in this case constant) is included as a next state variable is

$$\begin{aligned} \widehat{x}_{k+1} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \widehat{x}_k + \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} u_k \\ e_k &= \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \widehat{x}_k \end{aligned}$$

The system output is the tracking error. The weighting matrices Q and R are

$$\begin{aligned} &Q = e_k^T e_k = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \\ &R = 0.01 \end{aligned}$$

For this augmented state space model, the predictive controller presented in the Section 2 can be found. The augmented invariant set is in the Fig.4. The time-domain trajectories from simulations are shown in the Fig.5.

The maximal and minimal values of reference signal w_k^{max} and w_k^{min} at each sampling interval for which the system state does not leave the invariant set are piecewise affine functions of the system state x_k. These piecewise affine functions can be found off-line by solving the multi-parametric linear programs in MPT toolbox. The solution to this MP-LP are the partitions in the state space and in each partition, the minimal and maximal feasible values of reference signal are affine functions of the system state. The results for our example are in the Fig.6.

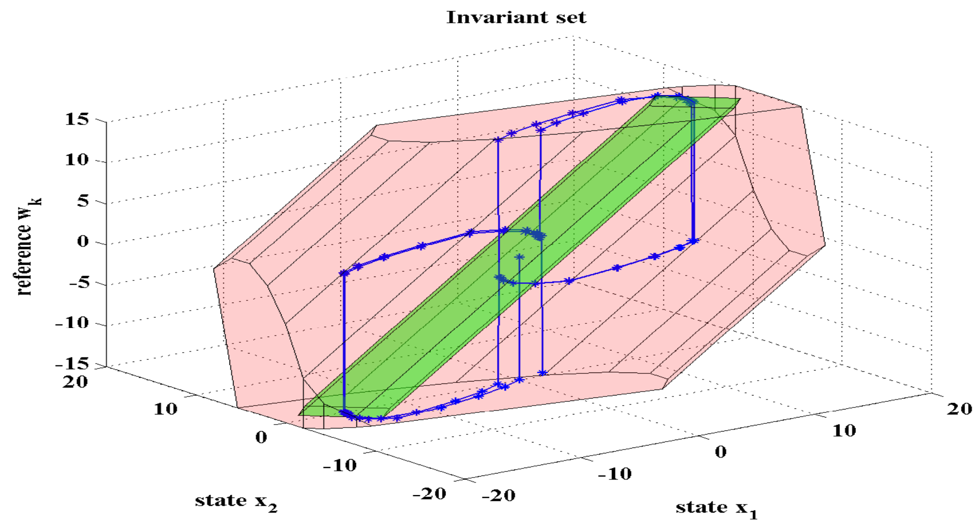

Figure 4: The augmented invariant polytopic sets.

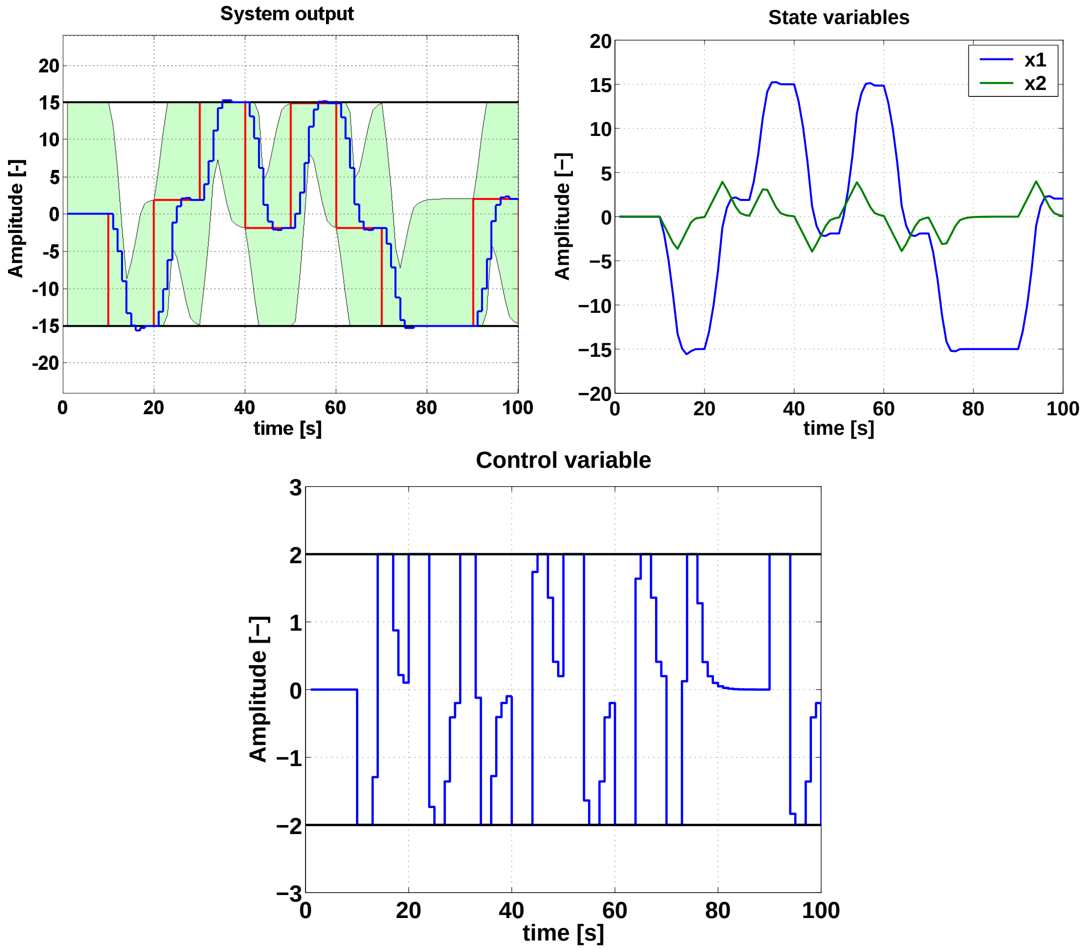

Figure 5: The simulation results.

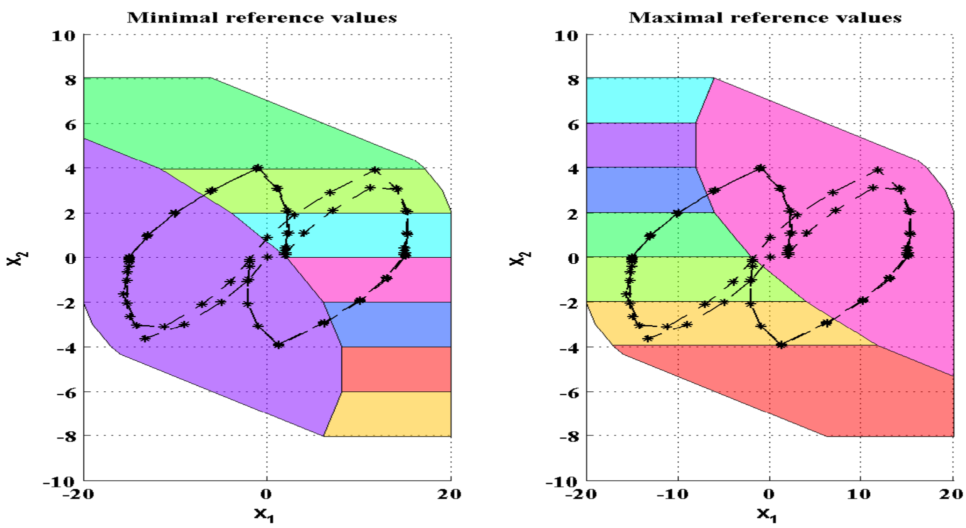

Figure 6: partitions of the solution to the multi-parametric linear program.

4 Conclusions

The stability of the control loops with model based predictive controllers are closely connected to the concept of invariant sets. In the basic form, this concept enables to control the system state to the origin of the state space. If there is a request the process output to follow some reference signal the basic concept must be extended. The reference tracking problem can be solved by augmenting the system state by reference signal.

MPC works with a finite prediction horizon and therefore the problem with feasibility can arise if a terminal set is used. In the tracking problem, the important question is how big or small the value of the set-point can be to ensure that the problem will be feasible for a given prediction horizon length.

In this paper, it was show, that the maximal and minimal bounds of the reference signal are piecewise affine functions of the system state. Because the results are the affine functions, computational demanding is minimal and therefore it is possible to check the feasibility of the problem for a given set-point on-line.

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