Simulation of Two-Dimensional Distributed Parameters Systems

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Abstract

The distributed parameters systems can be described by a linear two-dimensional (dependent on two spatial directions) parabolic partial differential equation. This paper deals with a transformation of this model to the classical linear dynamical state space model which can be used for control design. For accurate description, this model has a large dimension which can produce problems with advanced controller design, for example, with model predictive control approach. Therefore a model reduction has to be used for a controller design. The influence of the model reduction on the model accuracy is discussed.

Keywords: Distributed Parameters, Partial Differential Equations, Modelling, Reduced-order Models, Predictive Control.

1 INTRODUCTION

There are many industrial processes which have a distributed parameters behaviour. Consequently, these processes cannot be modelled by lumped inputs and lumped outputs models for correct representation.

This paper deals with two-dimensional dynamic processes (systems with parameters dependent on two spatial directions) which can be described by lumped inputs and distributed output models. These models can be mathematically described by partial differential equations (PDE) [2].

Unlike ordinary differential equations, the PDEs contain, in addition, a derivative with respect to the spatial directions. Consequently, the partial differential equations lead to a more accurate models but their complexity is larger.

The dynamic behaviour of the distributed parameters system, which is described by the PDE, can be approximately described by a finite-dimensional model, for example, by using the finite difference method [3]. Then the ordinary differential equation model with large dimension is obtained and can be used for a finite-dimensional controller design.

Unfortunately, for online solving a optimization problem, e.g. model predictive control approach, the large model dimension introduces a problem for the control design. Therefore a model reduction method is used and the influence of the model reduction on the model accuracy is presented.

The paper is organized as follows. In section 2, the distributed parameters model for the finite controller design is developed. In section 3, two model reduction methods are described and these model reduction methods are compared in a demonstration example in section 4.

2 DISTRIBUTED PARAMETERS PROCESS DESCRIPTION

In this section, a heat transfer process model, which is described by a two-dimensional linear parabolic PDE, is developed for the finite-dimensional controller design. At first, the stationary PDE is transformed to a linear equations system using the second order finite difference approximation [2]. Then the evolution partial differential equation is transformed to a linear discrete system using the implicit scheme [3].

2.1 Stationary Partial Differential Equation

For the surface thermal conductivity λ [W/K] independent on the temperature Θ [K] and a surface heat source $f(x,y)$ [W/m^2], the heat transfer process in the stationary case leads to the parabolic PDE

$$-\lambda\left(\frac{\partial^2\,\Theta(x,\,y)}{\partial\,x^2}+\frac{\partial^2\,\Theta(x,\,y)}{\partial\,y^2}\right) = \qquad (1)$$
$$= -\lambda\,\Delta\Theta(x,\,y) \;=\; f(x,\,y)\,.$$

Then the unknown temperature Θ must satisfy equation (1) on an open set $\Omega=(0,L_1)\times(0,L_2)$ and a boundary condition on $\partial\Omega$. Note that $\partial\Omega$ means the boundary of the set Ω.

In this paper, it is used the boundary condition which specify the temperature gradient on the boundary $\partial\Omega$ by the following statement

$$-\lambda\,\frac{\partial\,\Theta(x,y)}{\partial\,n}=\alpha\big(\Theta(x,y)-\Theta_s(x,y)\big) \qquad (2)$$

where n is the normal line, α [W/(mK)] is an external heat transfer coefficient and $\Theta_s(x,y)$ is the surrounding temperature. Note that equation (2) is known as Newton boundary condition or the boundary condition of the third kind [2].

For the transformation of the PDE (1) with the Newton boundary condition (2) to the finite-dimensional model, the set Ω is covered by an imaginary mesh so that the values of mesh points satisfy $\mathbf{\Theta}_{i,j}=\Theta(i\,\delta x,j\,\delta y)$ and $\mathbf{F}_{i,j}=f(i\,\delta x,j\,\delta y)$ where δx and δy are the grid sizes of the imaginary mesh and i, j are row and column indices respectively (see Figure 1). Matrix $\mathbf{\Theta}$ is the matrix of values of temperatures in the mesh points.

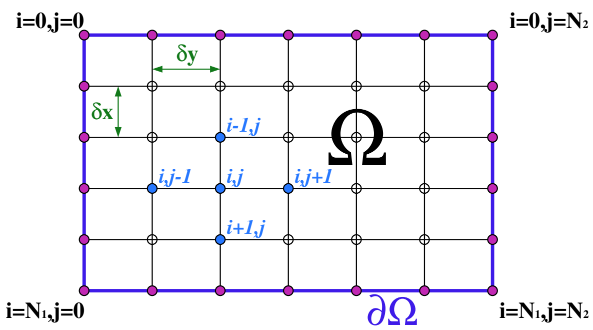

Figure 1: The mesh on the set Ω

Define ω as the set of all interior points of the set Ω

$$\omega=\Big\{(i\delta x,j\delta y)\,|\,i=1,2,\ldots,N_1-1;j=1,2,\ldots,N_2-1\Big\}$$

and $\bar\omega$ as the set of all points of the set Ω

$$\bar\omega=\Big\{(i\delta x,j\delta y)|i=0,1,\ldots,N_1;j=0,1,\ldots,N_2\Big\},$$

where the grid sizes are

$$\delta x=\frac{L_1}{N_1}\,,\qquad\qquad \delta x=\frac{L_2}{N_2}\,. \qquad (3)$$

Then the set of boundary points $\delta\omega$ is $\delta\omega=\bar\omega\setminus\omega$.

Using the second order difference approximation of the second derivative [3]

$$\frac{\partial^2\Theta}{\partial\,x^2}\approx\frac{\mathbf{\Theta}_{i+1,j}-2\mathbf{\Theta}_{i,j}+\mathbf{\Theta}_{i-1,j}}{\delta x^2}=\Delta_x\mathbf{\Theta}_{i,j}\,,$$
$$\frac{\partial^2\Theta}{\partial\,y^2}\approx\frac{\mathbf{\Theta}_{i,j+1}-2\mathbf{\Theta}_{i,j}+\mathbf{\Theta}_{i,j-1}}{\delta y^2}=\Delta_y\mathbf{\Theta}_{i,j}\,,$$

equation (1) can be written as the equation system

$$c_s\,\mathbf{\Theta}_{i+1,j}\;+\;c_e\,\mathbf{\Theta}_{i,j+1}+c_n\,\mathbf{\Theta}_{i-1,j}\;+ \qquad (4)$$
$$+\;c_w\,\mathbf{\Theta}_{i,j-1}+c_p\,\mathbf{\Theta}_{i,j}=\mathbf{F}_{i,j}\quad\text{on}\quad\omega\,,$$

where the coefficients are

$$c_s=c_n=-\frac{\lambda}{\delta x^2},\quad c_e=c_w=-\frac{\lambda}{\delta y^2},\quad c_p=\frac{2\lambda}{\delta x^2}+\frac{2\lambda}{\delta y^2}.$$

Using the first order difference approximation and elementary rearrangement, the Newton boundary condition (2) can be written as

$$\begin{aligned}
b_x\,\mathbf{\Theta}_{i,j}+b_n\,\mathbf{\Theta}_{i+1,j}&=\mathbf{\Theta}_{s_{i,j}}&&\text{for}\quad i=0\,,\\
b_x\,\mathbf{\Theta}_{i,j}+b_s\,\mathbf{\Theta}_{i-1,j}&=\mathbf{\Theta}_{s_{i,j}}&&\text{for}\quad i=N_1\,,\\
b_y\,\mathbf{\Theta}_{i,j}+b_e\,\mathbf{\Theta}_{i,j+1}&=\mathbf{\Theta}_{s_{i,j}}&&\text{for}\quad j=0\,,\\
b_y\,\mathbf{\Theta}_{i,j}+b_w\,\mathbf{\Theta}_{i,j-1}&=\mathbf{\Theta}_{s_{i,j}}&&\text{for}\quad j=N_2
\end{aligned} \qquad (5)$$

where $\mathbf{\Theta}_{s_{i,j}}=\Theta_s(i\,\delta x,j\,\delta y)$ and the coefficients are

$$b_s=b_n=-\frac{\lambda}{\alpha\,\delta x}\,,\quad b_e=b_w=-\frac{\lambda}{\alpha\,\delta y}\,,$$
$$b_x=1+\frac{\lambda}{\alpha\,\delta x}\,,\qquad b_y=1+\frac{\lambda}{\alpha\,\delta y}\,.$$

Then elements of the matrix $\mathbf{F}$ satisfy

$$\begin{aligned}
\mathbf{F}_{i,j}&=f(i\,\delta x,j\,\delta y)&&\text{on}\quad\omega\,,\\
\mathbf{F}_{i,j}&=\Theta_s(i\,\delta x,j\,\delta y)&&\text{on}\quad\delta\omega.
\end{aligned}$$

When we define vectors

$$\boldsymbol{\theta}=\begin{bmatrix}\mathbf{\Theta}(:,0)\\ \mathbf{\Theta}(:,1)\\ \vdots\\ \mathbf{\Theta}(:,N_2-1)\\ \mathbf{\Theta}(:,N_2)\end{bmatrix},\quad \mathbf{f}=\begin{bmatrix}\mathbf{F}(:,0)\\ \mathbf{F}(:,1)\\ \vdots\\ \mathbf{F}(:,N_2-1)\\ \mathbf{F}(:,N_2)\end{bmatrix},\qquad (6)$$

$$l=j(N_1+1)+i$$

where $\mathbf{\Theta}(\,:\,,0)$ means the zero column of the matrix $\mathbf{\Theta}$, $\mathbf{\Theta}(\,:\,,1)$ the first column and so on, the equation system (4) with equations (5) can be written in compact form

$$\mathbf{P}\,\boldsymbol{\theta} = \mathbf{f} \tag{7}$$

where the matrix $\mathbf{P}$ equals

$$\mathbf{P} = \begin{bmatrix} \bar{\mathbf{I}} & \tilde{\mathbf{I}} & & & & \\ \bar{\mathbf{D}} & \bar{\mathbf{I}}_n & \bar{\mathbf{D}} & & & \\ & \bar{\mathbf{D}} & \bar{\mathbf{I}}_n & \bar{\mathbf{D}} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \bar{\mathbf{D}} & \bar{\mathbf{I}}_n & \bar{\mathbf{D}} \\ & & & & \tilde{\mathbf{I}} & \bar{\mathbf{I}} \end{bmatrix}. \tag{8}$$

Note that all submatrices in the matrix (8) are square with dimension $(N_1+1)\times(N_1+1)$ and are equal to

$$\bar{\mathbf{I}}_n = \begin{bmatrix} b_x & b_n & & & & \\ c_n & c_p & c_s & & & \\ & c_n & c_p & c_s & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_n & c_p & c_s \\ & & & & b_s & b_x \end{bmatrix}, \quad \bar{\mathbf{D}} = \begin{bmatrix} 0 & & & & \\ & c_w & & & \\ & & c_w & & \\ & & & \ddots & \\ & & & & c_w & \\ & & & & & 0 \end{bmatrix},$$

$$\bar{\mathbf{I}} = \begin{bmatrix} \tilde{b} & \tilde{b}_s & & & & \\ & b_y & & & & \\ & & b_y & & & \\ & & & \ddots & & \\ & & & & b_y & \\ & & & & \tilde{b}_s & \tilde{b} \end{bmatrix}, \qquad \tilde{\mathbf{I}} = \begin{bmatrix} \tilde{b}_e & & & & \\ & b_w & & & \\ & & b_w & & \\ & & & \ddots & \\ & & & & b_w & \\ & & & & & \tilde{b}_e \end{bmatrix}$$

where

$$\begin{aligned} \tilde{b} &= 1 + \frac{\lambda}{2\alpha\delta x} + \frac{\lambda}{2\alpha\delta y}\,, \\ \tilde{b}_s &= -\frac{\lambda}{2\alpha\delta x}\,, \qquad \tilde{b}_e = -\frac{\lambda}{2\alpha\delta y}\,. \end{aligned}$$

2.2 Evolution Partial Differential Equation

For the thermal conductivity λ independent on the temperature Θ, the heat transfer process in the non-stationary (evolution) case leads to the following parabolic PDE

$$\rho\, c_0 \frac{d\Theta(x,y,t)}{dt} - \lambda\Delta\Theta(x,y,t) = f(x,y,t) \tag{9}$$

where ρ is the density of a medium and c_0 is its thermal capacity. Then the unknown temperature profile $\Theta(x,y,t)$, dependent on time t, must satisfy equation (9) on an open set Ω and the boundary condition on $\partial\Omega$ for all time horizon $\tilde{t} \in < t_0,\, t_{end} >$ and the initial condition $\Theta(x,y,t_0) = \Theta_{init}(x,y)$.

Using the implicit scheme [3] and the stationary PDE in the compact form (7), the evolution PDE (9) can be written as

$$\boldsymbol{\theta}(k+1) = \mathbf{M}\boldsymbol{\theta}(k) + \mathbf{N}\,\mathbf{f}(k), \qquad \boldsymbol{\theta}(t_0) = \boldsymbol{\theta}_{init}, \tag{10}$$

where matrices $\mathbf{M}$ and $\mathbf{N}$ equal

$$\mathbf{M} = \Big(\mathbf{I} + \frac{\delta t}{\rho\, c_0}\,\mathbf{P}\Big)^{-1}, \quad \mathbf{N} = \Big(\mathbf{I} + \frac{\delta t}{\rho\, c_0}\,\mathbf{P}\Big)^{-1} \frac{\delta t}{\rho\, c_0} \tag{11}$$

where $\mathbf{I}$ is the identity matrix with the corresponding dimension.

3 MODEL REDUCTION METHODS

The accuracy of the model (10) increases with decreasing grid sizes δx and δy. Unfortunately, for the advanced controller design, the low dimension model is needed. In this section, the model reduction by balanced truncation is shortly described.

3.1 Model Reduction by Balanced Truncation

There are infinitely many different state space realizations for a given transfer function. But some realizations are more useful in control design. One of these realizations is the balanced realization which gives balanced Gramians for controllability $\mathbf{W}_c$ and observability $\mathbf{W}_o$ [4]. In addition, these Gramians are equal to the diagonal matrix $\mathbf{\Sigma}$

$$\mathbf{W}_c = \mathbf{W}_o = \mathbf{\Sigma} = \mathrm{diag}\,(\sigma_1,\, \sigma_2,\, \ldots,\, \sigma_n)\,.$$

Note that the decreasingly order numbers,

$$\sigma_1 \;\geq\; \sigma_2 \;\geq\; \ldots \;\geq\; \sigma_n \;\geq 0,$$

are called the Hankel singular values of the system.

We suppose $\sigma_r \gg \sigma_{r+1}$ for some $r \in \langle 1; n)$. Then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ are less controllable and observable than those states corresponding to $\sigma_1, \ldots, \sigma_r$. The states corresponding to the singular values of $\sigma_{r+1}, \ldots, \sigma_n$ have smaller influence on the input-output behaviour of the system. Therefore, truncating those less controllable and observable states will not lose much information about the system and the dimension of the model can be very small.

3.2 Reduced Model for the Control Design

The reduced model for control of the evolution PDE (9) with the Newton boundary condition (2) can be written as

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\,\mathbf{x}(k) + \mathbf{B}\,\mathbf{u}(k) + \mathbf{E}\,\mathbf{z}(k), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \\ \mathbf{y}(k) &= \mathbf{C}\,\mathbf{x}(k) + \mathbf{D}\,\mathbf{u}(k) \end{aligned} \tag{12}$$

where $\mathbf{x}$ is a state of the model, $\mathbf{y}$ is its output, $\mathbf{u}$ is its input (manipulated variable), $\mathbf{z}$ represents the surrounding temperature and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ are state matrices.

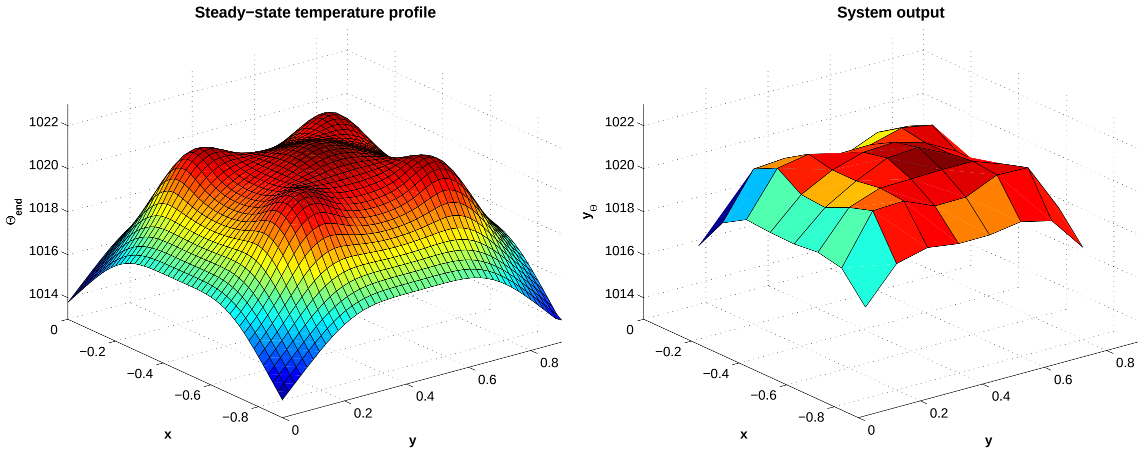

Figure 2: Steady-state temperature distribution $\Theta(x, y)$ and system output $\mathbf{y}$

4 DEMONSTRATION EXAMPLE

Consider a heat transfer process on the area $L_1 = L_2 = 0.9$ m which is described by the equation (9) with constants $\lambda = 51$ W/K, $\rho = 2500$ kg/m^2, $c_0 = 1259$ Ws/(kg K) and $\alpha = 1.14$ W/(mK). The grid sizes are $\delta x = \delta y = 0.02$ m and the sampling period is $T_s = 300$ s.

Consider that temperature of the heat transfer process is measured in 64 points which are uniform distributed over the area Ω.

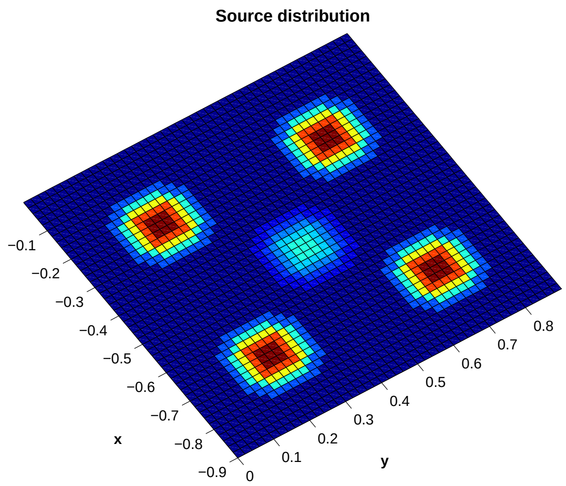

Figure 3: Heat source distribution $f(x, y)$

The heat source distribution $f(x, y)$ is shown in Figure 3. Figure 2a presents the steady-state temperature distribution for this heat source and the surrounding temperature $\Theta = 340$ K and Figure 2b shows the system output $\mathbf{y}$ – temperature in several measurement points.

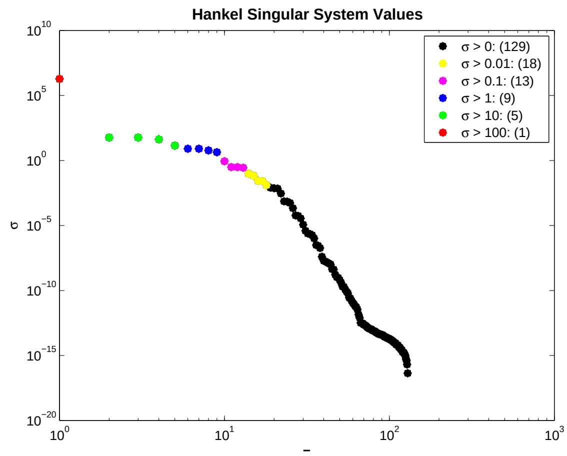

Figure 4: Hankel singular values of the system

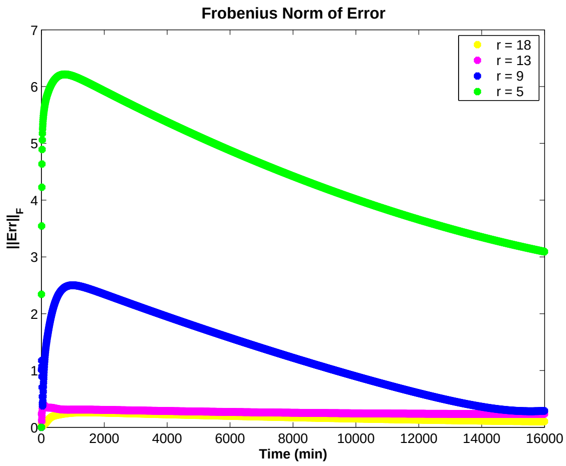

Figure 5: Frobenius norm of the model output error

Figure 4 shows the Hankel singular values of the system. From this figure follows that the system contains one singular value which is greater than 100 (red point in the figure), four singular values which are greater than 10 (red and green points) etc.

In this paper, the balanced truncation is used for $r \in \{18, 13, 9, 5\}$ states. Figure 5 shows the time response of Frobenius norm [1] of the model output. Note that the input signal of the system is unit step. From this figure follows that the Frobenius norm reaches a steady state for all reduced models.

5 CONCLUSION

The state space model of the distributed parameters system which is described by the linear two-dimensional parabolic partial differential equation is developed. Unfortunately, the dimension of this model is large and is not suitable for the advanced control design approach. Therefore the model reduction by balanced truncation is used. The models errors are compared and presented on demonstration example of the heat transfer process.

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